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**$(T^*\mathcal{B})_q$ ,  $q$ -analogue of model space  
and CGC generating matrices <sup>†</sup>**

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**Abstract**

We study relations between the deformed cotangent bundle  $(T^*\mathcal{B})_q$  for the Borel subgroup  $\mathcal{B}$  of a given simple Lie group  $G$ , the quantum Lie algebra  $\mathcal{J}_q$  associated with the corresponding quantum group  $G_q$  and the matrices generating Clebsch-Gordan coefficients (CGC) for  $\mathcal{J}_q$ . We reveal the connection of these objects to quantum analogue of the model space  $\mathcal{M}$  and  $q$ -tensor operators.

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# 1 INTRODUCTION.

Among different representations of a given compact Lie group  $G$  the model space  $\mathcal{M}$  plays a distinguished role. By definition,<sup>1</sup> the model space is a direct sum of all irreducible representations  $\mathcal{H}_j$  with multiplicity one

$$\mathcal{M} = \sum_j \oplus \mathcal{H}_j \quad (1.1)$$

realized in some universal way. A most popular form of  $\mathcal{M}$  is a space of holomorphic functions on the Borel subgroup  $\mathcal{B}$  of complexified form of the group  $G$ . In this construction the Borel subgroup is considered as an affine space.

A study of model spaces provides a natural language for investigation of physical models. For example, the popular model of 2-dimensional quantum gravity, introduced by Polyakov,<sup>2</sup> may be interpreted in terms of the model space of Virasoro algebra.<sup>3</sup> A finite-dimensional quantum group with deformation parameter, depending on the central charge, naturally appears in this context.

In the present paper, which was written with an intent to find new applications of model space in modern mathematical physics, we discuss a  $q$ -analogue of the model space related to  $q$ -deformed Lie group  $G_q$ . For this purpose we introduce and examine several "coordinatizations" of the quantum space  $(T^*\mathcal{B})_q$ . As a by-product we obtain some generating matrices for the set of Clebsch-Gordan coefficients (CGC). To our knowledge this result is new even for the non-deformed case.

Throughout the paper we systematically and intentionally make use of the  $R$ -matrix formalism, which we believe is the most convenient and powerful tool to get explicit results in the domain of quantum groups.

To avoid the known difficulties with compact forms of quantum groups we adopt here a convention to work with complexified objects (groups, algebras) and their finite dimensional representations on a formal algebraic level. We also do not discuss subtleties arising in the case of  $q$  being a root of unity.

Most of formulae given in this paper in  $R$ -matrix form have universal validity. However, the concrete results are illustrated on the simplest example  $G_q = SL_q(2)$ . The generalization to other groups needs more technical details such as an explicit structure of  $R$ -matrices and related objects.

Mentioned above "coordinatizations" of  $(T^*\mathcal{B})_q$  arise from two possible decompositions of the matrix  $L$  (in usual notations  $L = L_+ L_-^{-1}$ , it comprises all generators of the corresponding quantum Lie algebra) :

$$L = U D U^{-1} \quad \text{and} \quad L = A B A^{-1},$$

where  $D$  is a diagonal unimodular matrix,  $U$  is a deformation of unitary matrix,  $A$  and  $B$  are unimodular upper and lower triangular matrices. As we shall clarify below, the matrices  $A$  and  $B$  admit a natural interpretation as the coordinates in the base and in the fiber of  $(T^*\mathcal{B})_q$ , whereas entries of the matrix  $U$  will be shown to provide basic shifts

on the model space  $\mathcal{M}$  and generate  $q$ -analogues of Clebsch-Gordan coefficients for the quantum group  $G_q$ . The explicit connection between  $U$  and  $(A, B)$  will be demonstrated on the example of  $SL_q(2)$ .

It should be mentioned that an object like the matrix  $U$  appeared first in Refs.4,5 (later it was used also in Ref.6), where it was interpreted as a "chiral" component of the quantum group-like element  $g$ . In the present paper we give another interpretation and application of the matrix  $U$  in the context of a model space.

Let us briefly describe the contents of the present paper. In the Sec.II the definition of the cotangent bundle for a quantum group is reminded. Next we introduce an object of especial interest for us – the algebra  $\mathcal{U}$  generated by the entries of the matrix  $U$  which diagonalizes the coordinate in a fiber of  $(T^*G)_q$ . We derive explicit relations for this algebra in the case of  $G = SL(2)$ .

In the Sec.III we consider a non-deformed limit ( $q = 1$ ) of the algebra  $\mathcal{U}$  and construct an explicit representation. For the case of  $G = SL(2)$  we show that the matrix  $U_0$  generates Clebsch-Gordan coefficients (CGC) for the corresponding non-deformed Lie algebra. The Borel subgroup  $\mathcal{B}$  and the space  $T^*\mathcal{B}$  naturally appear here. Finally, we discuss a connection of our results with the Wigner-Eckart theorem.

In the Sec.IV we construct representations of the algebra  $\mathcal{U}$  (for  $q \neq 1$ ) for the case of  $SL(2)$  in two different ways. The first one uses the language of  $q$ -oscillators. The second is based on explicit realization of  $(T^*\mathcal{B})_q$  and hence involves a notion of quantum model space. Here we show that the matrix  $U$  is a "generating matrix" for CGC for deformed Lie algebra. We also give some comments on the generalized version of the Wigner-Eckart theorem.

## 2 $(T^*G)_q$ AND RELATED OBJECTS.

There exist three symplectic manifolds (from the physical point of view they are phase spaces) naturally related to a given Lie group  $G$  and its Lie algebra  $\mathcal{J}$ :

- ⟨1⟩  $T^*G$  – the cotangent bundle for the group  $G$ ;
- ⟨2⟩  $T^*\mathcal{B}$  – the cotangent bundle for the Borel subgroup  $\mathcal{B}$ ;
- ⟨3⟩  $\mathcal{O}$  – an orbit of the co-adjoint action of  $G$  on  $\mathcal{J}^*$ .

For instance, in the case of  $G = SL(2)$  (which will be our main example) these spaces are six-, four-, and two-dimensional, correspondingly.

The method of geometric quantization<sup>7</sup> provides a representation theory for ⟨1⟩, ⟨2⟩ and ⟨3⟩. Turning from classical to quantum groups, one can try to construct a representation theory for the deformed analogues of these manifolds. In the present paper we shall deal with deformations of the spaces ⟨1⟩ and ⟨2⟩.

### A. Description of $(T^*G)_q$ .

Let  $G_q$  be a deformation of the Lie group  $G$  and  $\mathcal{J}_q$  be a deformation of the corresponding Lie algebra  $\mathcal{J}$ . The deformed cotangent bundle  $(T^*G)_q$  is a non-commutative manifold, i.e., according to the ideology developed by A.Connes,<sup>8</sup> its coordinates are

(non-commuting) generators of some associative algebra. A point on this manifold is parameterized by the pair  $(g, L)$ , where  $g \in G_q$  is a coordinate in the base of the bundle, and  $L$  is a coordinate in a fiber.

The structure of  $(T^*G)_q$  is defined via commutation relations between the coordinates in the base and in a fiber. An appropriate  $R$ -matrix form of these relations was proposed in Ref.5:

$$R_{\pm} \overset{1}{g} \overset{2}{g} = \overset{2}{g} \overset{1}{g} R_{\pm}, \quad (2.1)$$

$$R_{-} \overset{1}{g} \overset{2}{L} = \overset{2}{L} R_{+} \overset{1}{g}, \quad (2.2)$$

$$\overset{1}{L} R_{-}^{-1} \overset{2}{L} R_{-} = R_{+}^{-1} \overset{2}{L} R_{+} \overset{1}{L}. \quad (2.3)$$

Here and below we use the formalism developed in Ref.9, i.e., objects like  $g$  and  $L$  are considered as matrices (say,  $L \in \mathcal{J}_q \otimes V$ , where  $V$  stands for auxiliary space). We use the standard notations for tensor products:  $\overset{1}{L} = L \otimes I \in \mathcal{J}_q \otimes V \otimes V$ , etc.

Let us take the parameter  $q$ , which appears in the theory of quantum groups, in the following form

$$q = e^{\gamma \hbar}, \quad (2.4)$$

where  $\hbar$  is the Planck constant (the parameter of quantization) and  $\gamma$  is the deformation parameter. In physical applications it is most natural to suppose that  $\gamma$  is either pure real ( $q$  belongs to the real axis) or pure imaginary ( $q$  belongs to the unit circle at the complex plane).

The second form of  $q$  is typical for the WZW theory.<sup>4,6,10</sup> For  $|q| = 1$  we suppose also that  $q$  is not a root of unity. It should be mentioned that for both variants of choice of  $\gamma$  in (2.4) the definition of  $q$ -number

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \quad (2.5)$$

is invariant with respect to complex conjugation of  $q$ , i.e.  $\overline{[x]} = [\bar{x}]$ . This property becomes important if one discusses involutions of deformed Lie algebras.

**Definition 1** *The algebra  $\mathcal{L}$  is an associative algebra generated by entries of the matrix  $L$  which obeys relation (2.3).*

An important fact – the connection of algebra  $\mathcal{L}$  with the corresponding quantum Lie algebra  $\mathcal{J}_q$  was established in Ref.11 in the following form.

**Proposition 1** *Let matrices  $L_+$ ,  $L_-$  obey the following exchange relations*

$$R_{\pm} \overset{1}{L_+} \overset{2}{L_+} = \overset{2}{L_+} \overset{1}{L_+} R_{\pm}, \quad R_{\pm} \overset{1}{L_-} \overset{2}{L_-} = \overset{2}{L_-} \overset{1}{L_-} R_{\pm}, \quad R_+ \overset{1}{L_+} \overset{2}{L_-} = \overset{2}{L_-} \overset{1}{L_+} R_+ . \quad (2.6)$$

*Then the matrix  $L = L_+ L_-^{-1}$  satisfies the relation (2.3).*

This statement implies that the algebra  $\mathcal{L}$  is isomorphic (up to some technical details which we do not discuss here) to corresponding quantum Lie algebra  $U_q(\mathcal{J})$  [which is defined by (2.6), see, e.g., Ref.9].

Consider now the relations (2.1)-(2.3) for  $g$  and  $L$  being  $2 \times 2$  matrices

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, \quad L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (2.7)$$

and the  $R$ -matrices taken in the form

$$R_+ = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \omega \equiv q - q^{-1}; \quad R_- = PR_+^{-1}P \quad (2.8)$$

( $P$  denotes the permutation operator:  $P \overset{1}{g} P = \overset{2}{g}$ , etc.). In this case (2.1)-(2.3) define the cotangent bundle for the quantum group  $G_q = GL_q(2)$ ; each of  $R$ -matrix equations (2.1) and (2.3) is equivalent to six independent relations:

$$\begin{aligned} q g_1 g_2 &= g_2 g_1, & q g_1 g_3 &= g_3 g_1, & q g_2 g_4 &= g_4 g_2, & q g_3 g_4 &= g_4 g_3, \\ g_2 g_3 &= g_3 g_2, & g_1 g_4 - q^{-1} g_4 g_1 &= -\omega g_2 g_3; \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} [A, B] &= -q^{-1}\omega BD, & [A, C] &= q^{-1}\omega DC, & [A, D] &= 0, \\ CD &= q^2 DC, & BD &= q^{-2} DB, & [B, C] &= q^{-1}\omega D(D - A). \end{aligned} \quad (2.10)$$

The equation (2.2) gives the following relations

$$\begin{aligned} g_1 A &= q A g_1 + \omega B g_3, & g_1 B &= B g_1, \\ g_2 A &= q A g_2 + \omega B g_4, & g_2 B &= B g_2, \\ g_3 A &= q^{-1} A g_3 + \omega g_1 C, & g_3 B &= B g_3 + \omega g_1 D, \\ g_4 A &= q^{-1} A g_4 + \omega g_2 C, & g_4 B &= B g_4 + \omega g_2 D, \\ g_1 C &= C g_1 + q^{-1}\omega D g_3, & g_1 D &= q^{-1} D g_1, \\ g_2 C &= C g_2 + q^{-1}\omega D g_4, & g_2 D &= q^{-1} D g_2, \\ g_3 C &= C g_3, & g_3 D &= q D g_3, \\ g_4 C &= C g_4, & g_4 D &= q D g_4. \end{aligned} \quad (2.11)$$

Next, let us recall the well-known statement (see, e.g., Ref.9):

**Proposition 2** *The algebra generated by the entries of the matrix  $g$  obeying (2.9) possesses the central element (“deformed determinant”)*

$$\det_q g = g_1 g_4 - q^{-1} g_2 g_3. \quad (2.12)$$

Similarly, for the algebra  $\mathcal{L}$  in the case of  $GL_q(2)$  one can check the following.

**Proposition 3** *The algebra with generators  $A, B, C, D$  obeying (2.10) possesses two central elements:*

$$K_1 = qA + q^{-1}D, \quad K_2 = q^{-1}AD - qBC. \quad (2.13)$$

Finally, using the commutation relations (2.11), one can check that

**Proposition 4** *The operators  $\det_q g$  and  $K_2$  commute with all entries of the matrices  $g$  and  $L$ .*

This implies that, fixing values of  $\det_q g$  and  $K_2$ , one gets a certain subalgebra of the algebra defined by (2.9)-(2.11).

**Definition 2** *Relations (2.9)-(2.11) for  $\det_q g = 1$  and  $K_2 = \text{const}$  define the cotangent bundle for the quantum group  $G_q = SL_q(2)$ .*

Let us underline that the above definitions and statements can be easily generalized, say to the case of  $SL_q(N)$ .

In our case the algebra  $\mathcal{L}$  is isomorphic to the quantum Lie algebra  $\mathcal{J}_q = U_q(sl(2))$  (introduced first in Ref.12) which is defined by the relations

$$[l_+, l_-] = \frac{q^{2l_3} - q^{-2l_3}}{q - q^{-1}} \equiv [2l_3], \quad q^{l_3} l_{\pm} = q^{\pm 1} l_{\pm} q^{l_3} \quad (2.14)$$

and the matrices  $L_{\pm}$  can be chosen as follows:

$$L_+ = \begin{pmatrix} q^{l_3} & \omega q^{1/2} l_- \\ 0 & q^{-l_3} \end{pmatrix}, \quad L_- = \begin{pmatrix} q^{-l_3} & 0 \\ -\omega q^{-1/2} l_+ & q^{l_3} \end{pmatrix}. \quad (2.15)$$

Note that the matrix  $L$  in the Proposition 1 is defined only up to a scaling factor. Thus, for  $L_+$ ,  $L_-$  given in (2.15), we may choose  $L$  as follows

$$L = q^2 L_+ L_-^{-1} = \begin{pmatrix} qC - q^{-2l_3} & q^{5/2} \omega l_- q^{-l_3} \\ q^{-1/2} \omega l_+ q^{-l_3} & q^2 q^{-2l_3} \end{pmatrix}. \quad (2.16)$$

Here  $C$  stands for the Casimir operator of  $U_q(sl(2))$  :

$$C = \omega^2 l_- l_+ + q^{2l_3+1} + q^{-(2l_3+1)} = q^{2\hat{j}+1} + q^{-(2\hat{j}+1)}, \quad (2.17)$$

where  $\hat{j}$  is the operator of spin.

According to Proposition 1, the matrix (2.16) satisfies (2.3). Therefore, it provides a (fundamental) representation of the algebra  $\mathcal{L}$  for  $U_q(sl(2))$ . In this representation the central elements (2.13) are given by

$$K_1 = q^2 C, \quad K_2 = q^3, \quad (2.18)$$

Note that the scaling factor  $q^2$  introduced in (2.16) has changed the values of  $K_1$  and  $K_2$ . The choice of such normalization in (2.16) will be explained later.

## B. Connection with quantum 6j-symbols.

Let us remind the theorem which describes an important property of the algebra  $\mathcal{L}$  for  $U_q(sl(2))$  (this statement appeared first in Ref.5).

**Theorem 1** Let  $D \equiv D(p)$  be the unimodular diagonal matrix

$$D = \begin{pmatrix} q^{p/\hbar} & \\ & q^{-p/\hbar} \end{pmatrix}, \quad (2.19)$$

and let  $2 \times 2$  matrix  $U$  satisfy the following exchange relations

$${}^1_1 \bar{D} {}^2_2 \bar{U} = {}^2_2 \bar{U} {}^1_1 \bar{D} \sigma, \quad {}^2_2 \bar{D} {}^1_1 \bar{U} = {}^1_1 \bar{U} {}^2_2 \bar{D} \sigma, \quad \sigma = \text{diag}(q^{-1}, q, q, q^{-1}), \quad (2.20)$$

$$R_+ {}^1_1 \bar{U} {}^2_2 \bar{U} = {}^2_2 \bar{U} {}^1_1 \bar{U} \mathcal{R}_+(p), \quad R_- {}^1_1 \bar{U} {}^2_2 \bar{U} = {}^2_2 \bar{U} {}^1_1 \bar{U} \mathcal{R}_-(p), \quad (2.21)$$

where  $R_{\pm}$  are the standard  $R$ -matrices (2.8) and

$$\mathcal{R}_+(p) = P \mathcal{R}_-^{-1}(p) P = q^{-1/2} \begin{pmatrix} q & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & \frac{q^{p/\hbar}}{[p/\hbar]} \\ -\frac{q^{-p/\hbar}}{[p/\hbar]} & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & q \end{pmatrix}, \quad (2.22)$$

(here  $[x]$  denotes a "q-number" (2.5)). Then matrix  $L$  constructed by means of the similarity transformation

$$L = U D U^{-1}, \quad (2.23)$$

satisfies the relation (2.3) and therefore its entries generate an algebra  $\mathcal{L}$  isomorphic to  $U_q(sl(2))$ .

The proof is given in appendix A. It makes use of the identity

$$\mathcal{R}_-(p) = (\bar{D})^{-1} \mathcal{R}_+(p) \sigma \bar{D}. \quad (2.24)$$

*Remark:* A consequence of (2.20) is the commutativity of  $L$  and  $D$

$${}^1_1 \bar{L} {}^2_2 \bar{D} = {}^2_2 \bar{D} {}^1_1 \bar{L}, \quad (2.25)$$

which implies that  $p$  commutes with all elements of  $\mathcal{L}$ . Later we shall interpret  $p$  as the operator of spin.

*Remark:* Properly generalizing objects which enter the Theorem 1, one can extend this theorem to the case of any quantum semisimple Lie algebra.<sup>13</sup> In particular, the matrix  $D$  for  $U_q(sl(N))$  is found to be :  $D(\vec{p}) = \text{const} \cdot q^{\vec{H} \otimes \vec{p}}$ , where  $\vec{p}$  consists of the operators corresponding to components of the weight vector (i.e., on each irreducible representation they are multiples of unity) and  $\vec{H}$  consists of the generators  $H_i$  of the Cartan subalgebra. An explicit form of  $\mathcal{R}(p)$  for  $U_q(sl(N))$  was obtained in Ref.14.

*Remark:* The matrix  $\mathcal{R}(p)$  obeys the deformed Yang-Baxter equation,<sup>14-16,5</sup> which can be written, for example, as follows :

$${}^1_{23} \bar{Q} \mathcal{R}_+(p) (\bar{Q})^{-1} {}^{13}_{23} \bar{Q} \mathcal{R}_+(p) {}^3_{12} \bar{Q} \mathcal{R}_+(p) (\bar{Q})^{-1} = {}^{12}_{23} \bar{Q} \mathcal{R}_+(p) {}^2_{13} \bar{Q} \mathcal{R}_+(p) (\bar{Q})^{-1} {}^{23}_{12} \bar{Q} \mathcal{R}_+(p), \quad (2.26)$$

where for  $\mathcal{J}_q = U_q(sl(2))$  the matrix  $Q = \begin{pmatrix} e^{i\xi} & \\ & e^{-i\xi} \end{pmatrix}$  contains an extra variable  $\xi$ , conjugated with  $p$ :

$$[p, \xi] = -i\hbar, \quad q^{p/\hbar} e^{i\xi} = q e^{i\xi} q^{p/\hbar}. \quad (2.27)$$

This variable  $\xi$  belongs to the algebra  $\mathcal{U}$  but does not enter matrix  $L$ . An explicit expression for  $\xi$  will be given below. The general form of  $Q$  for  $U_q(sl(N))$  can be easily found<sup>13</sup> :  $Q = e^{i\vec{H} \otimes \vec{\xi}}$ , where components of  $\xi_i$  are operators conjugated to  $p_i$  :  $[p_j, \xi_k] = -i\hbar \delta_{jk}$ .

The matrix  $\mathcal{R}(p)$  was discussed in physical literature in different contexts. In particular, it plays significant role in studies of quantum Liouville<sup>15,16</sup> and WZW<sup>4-6</sup> models; its relation to Calogero-Moser model was recently discussed in Ref.17. But for us more important fact is a connection of  $\mathcal{R}(p)$  with the quantum  $6j$ -symbols: the entries of (2.22) calculated on irreducible representations coincide (up to some normalization) with the values of some  $6j$ -symbols for  $U_q(sl(2))$  (exact formulae are given in Ref.18, generalizations are discussed in Ref.13). This connection allows to assume that objects like the matrix  $U$  should be interpreted in terms of Clebsch-Gordan coefficients (CGC). Below we demonstrate that  $U$  is indeed a “generating matrix” for CGC and clarify its relation to  $(T^*\mathcal{B})_q$ .

### C. Algebra $\mathcal{U}$ .

**Definition 3** *The algebra  $\mathcal{U}$  is an associative algebra generated by entries of matrix  $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$  and the operator  $p$  such that relations (2.19)-(2.22) hold.*

*Remark:* For simplicity we restricted our consideration to the case of  $\mathcal{U}$  associated with  $U_q(sl(2))$ . Let us stress that the case of  $\mathcal{U}$  associated with  $U_q(sl(N))$  can be studied similarly but it will involve more technical details. On the other hand, it might be rather cumbersome to obtain exact formulae for  $\mathcal{U}$  associated with  $U_q(\mathcal{J})$  in the case of  $\mathcal{J}$  being generic semisimple Lie algebra.

Let us give an explicit form of the defining relations (2.21) :

$$U_1 U_3 = q^{-1} U_3 U_1, \quad U_2 U_4 = q^{-1} U_4 U_2, \quad (2.28)$$

$$U_1 U_2 = U_2 U_1 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar + 1]}}, \quad U_3 U_4 = U_4 U_3 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar + 1]}} \quad (2.29)$$

$$U_1 U_4 = U_4 U_1 \frac{\sqrt{[p/\hbar + 1][p/\hbar - 1]}}{[p/\hbar]} - U_3 U_2 \frac{q^{p/\hbar}}{[p/\hbar]} \quad (2.30)$$

$$U_3 U_2 = U_2 U_3 \frac{\sqrt{[p/\hbar + 1][p/\hbar - 1]}}{[p/\hbar]} - U_1 U_4 \frac{q^{-p/\hbar}}{[p/\hbar]} \quad (2.31)$$

The rest of the relations contained in (2.21) are not independent and can be deduced from (2.28)-(2.31).

Additionally, from (2.20) one gets

$$\begin{aligned} q^{p/\hbar} U_1 &= q^{-1} U_1 q^{p/\hbar}, & q^{p/\hbar} U_2 &= q U_2 q^{p/\hbar}, \\ q^{p/\hbar} U_3 &= q^{-1} U_3 q^{p/\hbar}, & q^{p/\hbar} U_4 &= q U_4 q^{p/\hbar}. \end{aligned} \quad (2.32)$$



Thus, relations (2.28)-(2.32) describe the algebra  $\mathcal{U}$ . Using them, one may verify the following statement:

**Proposition 5** *A central element of  $\mathcal{U}$  is given by the "deformed" determinant of the matrix  $U$ :*

$$\text{Det}U \equiv U_1 U_4 \sqrt{\frac{[p/\hbar + 1]}{[p/\hbar]}} - U_2 U_3 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar]}} = q U_4 U_1 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar]}} - q U_3 U_2 \sqrt{\frac{[p/\hbar + 1]}{[p/\hbar]}}. \quad (2.33)$$

For fixed value of  $\text{Det}U$  the algebra  $\mathcal{U}$  contains only four independent generators. In classical limit ( $\hbar = 0$ ) they become the coordinates on 4-dimensional phase space.

For further discussion it is convenient to introduce new variables instead of  $U_i$ :

$$\hat{U}_i = U_i \sqrt{[p/\hbar]}. \quad (2.34)$$

The coordinates  $\{p, \hat{U}_i\}$  form a new set of generators of the algebra  $\mathcal{U}$ . The commutation relations (2.28)-(2.32) rewritten in terms of the new generators acquire a simpler form:

$$\hat{U}_1 \hat{U}_3 = q^{-1} \hat{U}_3 \hat{U}_1, \quad \hat{U}_2 \hat{U}_4 = q^{-1} \hat{U}_4 \hat{U}_2, \quad \hat{U}_1 \hat{U}_2 = \hat{U}_2 \hat{U}_1, \quad \hat{U}_3 \hat{U}_4 = \hat{U}_4 \hat{U}_3 \quad (2.35)$$

$$\hat{U}_1 \hat{U}_4 = \hat{U}_4 \hat{U}_1 \frac{[p/\hbar + 1]}{[p/\hbar]} - \hat{U}_3 \hat{U}_2 \frac{q^{p/\hbar}}{[p/\hbar]}, \quad (2.36)$$

$$\hat{U}_3 \hat{U}_2 = \hat{U}_2 \hat{U}_3 \frac{[p/\hbar + 1]}{[p/\hbar]} - \hat{U}_1 \hat{U}_4 \frac{q^{-p/\hbar}}{[p/\hbar]}, \quad (2.37)$$

$$\begin{aligned} q^{p/\hbar} \hat{U}_1 &= q^{-1} \hat{U}_1 q^{p/\hbar}, & q^{p/\hbar} \hat{U}_2 &= q \hat{U}_2 q^{p/\hbar}, \\ q^{p/\hbar} \hat{U}_3 &= q^{-1} \hat{U}_3 q^{p/\hbar}, & q^{p/\hbar} \hat{U}_4 &= q \hat{U}_4 q^{p/\hbar}. \end{aligned} \quad (2.38)$$

The central element (2.33) in new variables looks as follows

$$\text{Det}U \equiv (\hat{U}_1 \hat{U}_4 - \hat{U}_2 \hat{U}_3) \frac{1}{[p/\hbar]} = (\hat{U}_4 \hat{U}_1 - \hat{U}_3 \hat{U}_2) \frac{q}{[p/\hbar]}. \quad (2.39)$$

The explicit form of the matrix inverse to  $\hat{U}$ , which we shall need later, is

$$\hat{U}^{-1} = \frac{1}{\text{Det}U} \begin{pmatrix} \hat{U}_4 & -q\hat{U}_2 \\ -\hat{U}_3 & q\hat{U}_1 \end{pmatrix} \frac{1}{[p/\hbar]}. \quad (2.40)$$

Finally, from (2.34) we conclude that the expression (2.23) for the matrix  $L$  looks similarly in terms of new matrix  $\hat{U}$ :

$$L = U D U^{-1} = \hat{U} D \hat{U}^{-1}. \quad (2.41)$$

### 3 NON-DEFORMED CASE.

#### A. Representation of algebra $\mathcal{U}_0$ .

First, we consider the limit  $\gamma \rightarrow 0$ ,  $\hbar \neq 0$  (note that  $q$ -numbers turn into ordinary numbers), i.e., here we deal with a well understood situation – the representation theory of  $SL(2)$ . An investigation of this simple non-deformed case will make further results more transparent.

Let us denote the corresponding limit algebra as  $\mathcal{U}_0$ . The defining  $R$ -matrix relations (2.21) now degenerate to

$${}^1_0 U_0 {}^2_0 U_0 = {}^2_0 U_0 {}^1_0 U_0 \mathcal{R}_{\pm}^0(p), \quad (3.1)$$

where

$$\mathcal{R}_{+}^0(p) = \mathcal{R}_{-}^0(p) = \begin{pmatrix} 1 & \frac{\sqrt{(p/\hbar + 1)(p/\hbar - 1)}}{(p/\hbar)} & \frac{\hbar}{p} \\ -\frac{\hbar}{p} & \frac{\sqrt{(p/\hbar + 1)(p/\hbar - 1)}}{(p/\hbar)} & 1 \end{pmatrix}. \quad (3.2)$$

The analogues of relations (2.35)-(2.38) for  $\mathcal{U}_0$  are (from now on we omit the index 0 for the generators of  $\mathcal{U}_0$ )

$$p\hat{U}_1 = \hat{U}_1(p - \hbar), \quad p\hat{U}_2 = \hat{U}_2(p + \hbar), \quad p\hat{U}_3 = \hat{U}_3(p - \hbar), \quad p\hat{U}_4 = \hat{U}_4(p + \hbar), \quad (3.3)$$

$$[\hat{U}_1, \hat{U}_2] = [\hat{U}_1, \hat{U}_3] = [\hat{U}_2, \hat{U}_4] = [\hat{U}_3, \hat{U}_4] = 0, \quad (3.4)$$

$$[\hat{U}_1, \hat{U}_4] = \text{Det}U_0, \quad [\hat{U}_3, \hat{U}_2] = -\text{Det}U_0, \quad (3.5)$$

where  $\text{Det}U_0$  stands for a limit version of (2.39):

$$\text{Det}U_0 = (\hat{U}_1\hat{U}_4 - \hat{U}_2\hat{U}_3)\frac{\hbar}{p} = (\hat{U}_4\hat{U}_1 - \hat{U}_3\hat{U}_2)\frac{\hbar}{p}. \quad (3.6)$$

**Proposition 6** *A possible solution for (3.3)-(3.6) is*

$$\hat{U}_1 = \partial_1, \quad \hat{U}_2 = z_2, \quad \hat{U}_3 = -\partial_2, \quad \hat{U}_4 = z_1; \quad (3.7)$$

$$p = \hbar(z_1\partial_1 + z_2\partial_2 + 1), \quad (3.8)$$

where we denote  $\partial_i \equiv \frac{\partial}{\partial z_i}$ .

*Remark:* The representation given by (3.7)-(3.8) is not unique. In particular, the rescaling  $\hat{U}_i \rightarrow c_i \hat{U}_i$  (where  $c_i$  are numerical constants such that  $c_1 c_4 = c_2 c_3$ ) is allowable.

The Proposition 6 together with the connection formula (2.34) allows us to write out the explicit form of the matrix  $U_0$

$$U_0 = \begin{pmatrix} \partial_1 & z_2 \\ -\partial_2 & z_1 \end{pmatrix} \sqrt{\frac{\hbar}{p}}. \quad (3.9)$$

Note that this matrix is “unimodular”, i.e.,  $\text{Det}U_0 = (\partial_1 z_1 + z_2 \partial_2) \frac{\hbar}{p} = 1$ .

To describe the obtained representation of the algebra  $\mathcal{U}_0$  completely one has to define a space where operators (3.7)-(3.9) act. It is natural to think that this space is  $D(z_1, z_2)$  – a space of holomorphic functions of two complex variables.

Let us recall that  $D(z_1, z_2)$  is a space spanned on the vectors

$$|j, m\rangle = \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad m = -j, \dots, j \quad (3.10)$$

and equipped with the scalar product

$$\langle f, g \rangle = \frac{1}{(2\pi i)^2} \int \overline{f(z_1, z_2)} g(z_1, z_2) e^{-z_1 \bar{z}_1 - z_2 \bar{z}_2} dz_1 d\bar{z}_1 dz_2 d\bar{z}_2. \quad (3.11)$$

The system (3.10) is orthonormal with respect to the scalar product (3.11), that is  $\langle j, m | j', m' \rangle = \delta_{jj'} \delta_{mm'}$ . For the given scalar product a rule of conjugation of operators looks as follows

$$(z_i)^* = \partial_i, \quad (\partial_i)^* = z_i. \quad (3.12)$$

The question concerning unitarity of the matrix  $U_0$  is discussed in Appendix B.

## B. Connection with $\mathbf{T^*B}$ .

The generators of  $sl(2)$  can be realized on  $D(z_1, z_2)$  as differential operators:

$$l_+ = z_1 \partial_2, \quad l_- = z_2 \partial_1, \quad l_3 = \frac{1}{2} (z_1 \partial_1 - z_2 \partial_2). \quad (3.13)$$

Using these expressions we can compare the representation of the algebra  $\mathcal{L}$  (or, more precisely, its limit version  $\mathcal{L}_0$ ) given by Theorem 1 with the representation given by Proposition 1.

Indeed, in the limit  $\gamma \rightarrow 0$  the initial formula (2.23) acquires form

$$L = I + \gamma L_0 + O(\gamma^2), \quad L_0 = U_0 \begin{pmatrix} p & \\ & -p \end{pmatrix} U_0^{-1}. \quad (3.14)$$

Substituting here the explicit expressions (3.8)-(3.9) for  $p$ ,  $U_0$  and using the representation (3.13) for generators of  $sl(2)$ , one derives the following limit form of the  $L$ -operator:

$$L_0 = \hbar \begin{pmatrix} 2 + z_1 \partial_1 - z_2 \partial_2 & 2z_2 \partial_1 \\ 2z_1 \partial_2 & 2 - z_1 \partial_1 + z_2 \partial_2 \end{pmatrix} = 2\hbar \begin{pmatrix} 1 + l_3 & l_- \\ l_+ & 1 - l_3 \end{pmatrix}. \quad (3.15)$$

Notice that (3.15) exactly coincides with (2.16) taken in the limit  $\gamma \rightarrow 0$ . This explains why we had to introduce the factor  $q^2$  in (2.16).

The next observation concerning the limit of  $L$ -operator reads as follows.

**Proposition 7** *The matrix  $L_0$  in the representation (3.15) admits the decomposition*

$$L_0 = A_0 B_0 A_0^{-1}, \quad (3.16)$$

where

$$A_0 = \begin{pmatrix} z_1^{-1/2} & -z_1^{-1/2} z_2 \\ 0 & z_1^{1/2} \end{pmatrix}, \quad B_0 = \hbar \begin{pmatrix} p/\hbar + 1/2 & 0 \\ 2\partial_2 & -(p/\hbar - 1/2) \end{pmatrix} \quad (3.17)$$

and  $p$  is defined as in (3.8).

This statement can be verified directly.

Let us comment on the meaning of this proposition. First, note that  $A_0$  is a realization of a group-like element of the Borel subgroup of  $SL(2)$ . Moreover, this explicit form of  $A_0$  is straightly connected with the construction of the model space  $\mathcal{M}$  developed by Gelfand et al.<sup>1</sup> Indeed, the space  $D(z_1, z_2)$  being a realization of the model space for  $SL(2)$  (compare (1.1) and (3.10)) is spanned on monomials with arguments which are combinations of the entries of  $A_0$ . On the other hand,  $B_0$  is of opposite (with respect to  $A_0$ ) triangularity and its entries are operators acting on a given realization of the model space. Therefore,  $B_0$  can be regarded as an element of the space dual to the corresponding Borel subalgebra.

Thus,  $A_0$  and  $B_0$  are coordinates in the base and in a fiber of the cotangent bundle  $T^*\mathcal{B}$ . At this stage the appearance of  $T^*\mathcal{B}$  "inside" the algebra  $\mathcal{L}$  looks somewhat mysterious, but we shall clarify it later.

### C. Clebsch-Gordan coefficients.

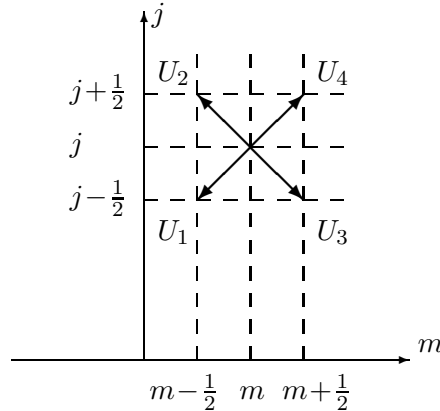
Let us consider an action of the generators of the algebra  $\mathcal{U}_0$  defined in (3.8)-(3.9) on the space  $D(z_1, z_2)$  (which is a realization of the model space). The action of these operators on the basic vectors (3.10) is given by

$$p |j, m\rangle = (2j + 1)\hbar |j, m\rangle, \quad (3.18)$$

$$\begin{aligned} U_1 |j, m\rangle &= \left(\frac{j+m}{2j+1}\right)^{1/2} |j - \frac{1}{2}, m - \frac{1}{2}\rangle, & U_2 |j, m\rangle &= \left(\frac{j-m+1}{2j+1}\right)^{1/2} |j + \frac{1}{2}, m - \frac{1}{2}\rangle, \\ U_3 |j, m\rangle &= -\left(\frac{j-m}{2j+1}\right)^{1/2} |j - \frac{1}{2}, m + \frac{1}{2}\rangle, & U_4 |j, m\rangle &= \left(\frac{j+m+1}{2j+1}\right)^{1/2} |j + \frac{1}{2}, m + \frac{1}{2}\rangle. \end{aligned} \quad (3.19)$$

Formula (3.18) allows us to identify the operator  $p$  as  $p = 2\hat{j} + 1$ , where  $\hat{j}$  is the operator of spin. Hence, invariant subspaces of  $p$  on the model space are those with fixed value of spin  $j$ .

Formulae (3.19) show that  $U_i$  are generators of the basic shifts on the model space (as illustrated on Fig.1). This observation is very important. As we shall see later, the same picture holds for  $q \neq 1$ .



**Fig.1:** Action of the operators  $U_i$  on the model space.

Now comparing the matrix elements  $\langle j'', m'' | U_i | j, m \rangle$  following from (3.19) with values of the Clebsch-Gordan coefficients (CGC) for decomposition of the tensor product of irreducible representations  $V_j \otimes V_{\frac{1}{2}}$  for  $sl(2)$  which are given by the Van-der-Waerden formula

$$\left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & m' & m'' \end{array} \right\} = \delta_{m'', m+m'} \sqrt{\frac{(j + \frac{1}{2} - j'')!(j + j'' - \frac{1}{2})!(j'' + \frac{1}{2} - j)!}{(j + j'' + \frac{3}{2})!}} \times \quad (3.20)$$

$$\times \sum_{r \geq 0} \frac{(-1)^r \sqrt{(j+m)!(j-m)!(j''+m'')!(j''-m'')!(2j''+1)}}{r!(j + \frac{1}{2} - j'' - r)!(j - m - r)!(\frac{1}{2} + m' - r)!(j'' - \frac{1}{2} + m + r)!(j'' - j - m' + r)!},$$

we establish the following correspondence

$$\begin{aligned} \langle j'', m'' | U_1 | j, m \rangle &= \delta_{j'', j-\frac{1}{2}} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{array} \right\}, \\ \langle j'', m'' | U_2 | j, m \rangle &= \delta_{j'', j+\frac{1}{2}} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{array} \right\}, \\ \langle j'', m'' | U_3 | j, m \rangle &= \delta_{j'', j-\frac{1}{2}} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{array} \right\}, \\ \langle j'', m'' | U_4 | j, m \rangle &= \delta_{j'', j+\frac{1}{2}} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{array} \right\}. \end{aligned} \quad (3.21)$$

Thus, we proved the following statement:

**Proposition 8** *The generators  $U_i$  of the algebra  $\mathcal{U}_0$  are operators of the basic shifts on the model space for  $sl(2)$  and they generate the Clebsch-Gordan coefficients corresponding to decomposition of the product  $V_j \otimes V_{\frac{1}{2}}$  of the irreps of  $sl(2)$ .*

This statement allows to call the matrix  $U_0$  a "generating matrix" (by analogy with the notion of a generating function) for CGC.

*Remark:* Usually, introducing a generating object (well-known examples are the generating functions for different sets of polynomials, e.g., for the Legendre polynomials), one makes properties of the objects under consideration more evident. We think that the notion of generating matrix will be useful for calculations involving CGC of classical and quantum algebras.

#### D. Wigner-Eckart theorem.

One should underline a connection of the results obtained above (Proposition 8) and the well-known mathematical construction – Wigner-Eckart theorem,<sup>19</sup> which has important applications in quantum mechanics.

Let us remind that the Wigner-Eckart theorem gives CGC for classical Lie algebra  $\mathcal{J}$  as matrix elements of some set of operators. These operators are called *tensor operators*. They map the corresponding model space  $\mathcal{M}$  onto itself and have special transformation properties under adjoint action of the algebra. In the case of  $\mathcal{J} = sl(2)$  the Wigner-Eckart theorem reads as follows.

**Theorem 2** Let  $l_+$ ,  $l_-$  and  $l_3$  be the generators of  $sl(2)$  and let  $T_m^j$ ,  $m = -j, \dots, j$  be a set of operators acting on  $\mathcal{M}$  and obeying the commutation relations

$$[l_3, T_m^j] = m T_m^j, \quad [l_{\pm}, T_m^j] = \sqrt{(j \mp m)(j \pm m + 1)} T_{m \pm 1}^j, \quad (3.22)$$

where  $j(j+1)$  is an eigenvalue of the Casimir operator for  $sl(2)$ . Then the matrix elements of  $T_m^j$  on  $\mathcal{M}$  are proportional to Clebsch-Gordan coefficients:

$$\langle j'' m'' | T_m^j | j' m' \rangle = C_{j'' m''}^j \left\{ \begin{matrix} j' & j & j'' \\ m' & m & m'' \end{matrix} \right\},$$

where the coefficients  $C_{j'' m''}^j$  do not depend on  $m$ ,  $m'$ ,  $m''$ .

Proposition 8 says that any tensor operators of spin  $j = 1/2$  (that is  $\{T_{1/2}^{1/2}, T_{-1/2}^{1/2}\}$ ,  $T_m^{1/2} : V_j \mapsto V_j \otimes V_{1/2} = V_{j+1/2} \oplus V_{j-1/2}$ ) may be constructed via the operators  $U_i$  (in fact, it is evident from Fig.1). Indeed, comparing the commutation relations obtained directly from (3.9) and (3.13)

$$\begin{aligned} [l_+, U_1] &= U_3, & [l_+, U_2] &= U_4, & [l_+, U_3] &= 0, & [l_+, U_4] &= 0, \\ [l_-, U_1] &= 0, & [l_-, U_2] &= 0, & [l_-, U_3] &= U_1, & [l_-, U_4] &= U_2, \\ [l_3, U_1] &= -\frac{1}{2}U_1, & [l_3, U_2] &= -\frac{1}{2}U_2, & [l_3, U_3] &= \frac{1}{2}U_3, & [l_3, U_4] &= \frac{1}{2}U_4 \end{aligned} \quad (3.23)$$

with Theorem 2, we get the following.

**Proposition 9** The generators  $U_i$  of the algebra  $\mathcal{U}_0$  form a basis for tensor operators of spin  $1/2$ , that is components  $T_{1/2}^{1/2}$  and  $T_{-1/2}^{1/2}$  of any tensor operator of spin  $1/2$  can be realized as linear combinations of  $U_i$ :

$$T_{-1/2}^{1/2} = \mu(p) U_1 + \nu(p) U_2, \quad T_{1/2}^{1/2} = \mu(p) U_3 + \nu(p) U_4, \quad (3.24)$$

where  $\mu(p)$  and  $\nu(p)$  are functions only of  $p = \hat{2j} + 1$ .

## 4 DEFORMED CASE.

Now we want to extend the results obtained in the previous section to the case of  $q \neq 1$ . In particular, we are going to examine the representations of the algebra  $\mathcal{U}$  (see Definition 3 above) and to show that the corresponding matrix  $U$  generates Clebsch-Gordan coefficients for the deformed Lie algebra. For these purposes we shall exploit a natural connection of  $\mathcal{U}$  with  $(T^* \mathcal{B})_q$ .

### A. The q-oscillators approach.

There exist different ways to obtain desirable representations of the algebra  $\mathcal{U}$ . First we describe a more direct but less instructive method, which is similar to that used in the non-deformed case.

By analogy with the non-deformed case studied above, one can assume that the entries of the matrix  $U$  might be realized as operators (deformations of those obtained in Proposition 6) acting on the space of two complex variables. Indeed, using the definition (2.33) of the central element of  $\mathcal{U}$  and taking into account the identity for  $q$ -numbers

$$[a] q^b + [b] q^{-a} = [a + b], \quad (4.1)$$

we can rewrite (2.35)-(2.37) in the following way:

$$\widehat{U}_1 \widehat{U}_3 = q^{-1} \widehat{U}_3 \widehat{U}_1, \quad \widehat{U}_2 \widehat{U}_4 = q^{-1} \widehat{U}_4 \widehat{U}_2, \quad \widehat{U}_1 \widehat{U}_2 = \widehat{U}_2 \widehat{U}_1, \quad \widehat{U}_3 \widehat{U}_4 = \widehat{U}_4 \widehat{U}_3, \quad (4.2)$$

$$\widehat{U}_1 \widehat{U}_4 - q^{-1} \widehat{U}_4 \widehat{U}_1 = q^{-1} \text{Det} U q^{p/\hbar}, \quad \widehat{U}_3 \widehat{U}_2 - q \widehat{U}_2 \widehat{U}_3 = -\text{Det} U q^{-p/\hbar}. \quad (4.3)$$

The relations (4.3) are well known in the theory of  $q$ -oscillators ( $q$ -bosons).<sup>20</sup> Recall that  $q$ -analogues of creation, annihilation, and number operators form the deformed Heisenberg algebra defined by the commutation relations

$$a a^+ - q a^+ a = N^{-1}, \quad N a = q^{-1} a N, \quad N a^+ = q a^+ N, \quad (4.4)$$

and they can be realized in terms of multiplication and difference operators:

$$a^+ = z, \quad a = z^{-1} [z \partial_z], \quad N = q^{z \partial_z}. \quad (4.5)$$

Using two pairs of generators of the deformed Heisenberg algebra, one can construct the generators of  $U_q(sl(2))$ :  $l_+ = a_1^+ a_2$ ,  $l_- = a_2^+ a_1$ ,  $q^{l_3} = N_1^{1/2} N_2^{-1/2}$ . Applying here the representation (4.5) one gets

$$l_+ = z_1 z_2^{-1} [z_2 \partial_2], \quad l_- = z_2 z_1^{-1} [z_1 \partial_1], \quad q^{l_3} = q^{\frac{1}{2}(z_1 \partial_1 - z_2 \partial_2)}. \quad (4.6)$$

The Casimir operator (2.17) of  $U_q(sl(2))$  in this realization is given by

$$C = q N_1 N_2 + q^{-1} N_1^{-1} N_2^{-1}. \quad (4.7)$$

Now, comparing, (4.2)-(4.3) with (4.4), it is easy to conclude that the pairs  $(\widehat{U}_1, \widehat{U}_4)$  and  $(\widehat{U}_2, \widehat{U}_3)$  are similar to two pairs of  $q$ -boson operators.

Taking into account the Weyl-like form of relations (4.2) and having already found explicit expressions (3.7)-(3.8) for the generators of algebra  $\mathcal{U}_0$ , we get an answer for  $D$  and  $\widehat{U}$  in terms of  $q$ -oscillators. More precisely, a straightforward calculation allows to verify the following statement:

**Proposition 10** *Equations (4.2)-(4.3) have the family of solutions:*

$$q^{p/\hbar} = q N_1 N_2, \quad \widehat{U} = \begin{pmatrix} \alpha_0 a_1 N_1^\alpha N_2^{-\beta} & \beta_0 a_2^+ N_1^\beta N_2^{-\alpha} \\ -\gamma_0 a_2 N_1^{-(1+\beta)} N_2^\alpha & \delta_0 a_1^+ N_1^{-\alpha} N_2^{1+\beta} \end{pmatrix}, \quad (4.8)$$

where  $\alpha_0 \delta_0 = q \beta_0 \gamma_0$ .

Let us note that this form of  $\widehat{U}$  is consistent with the condition (2.32).

Taking into account the connection formula (2.34) and applying to the generators  $a_i$ ,  $a_i^+$ ,  $N_i$  the representation (4.5), one obtains from (4.8) a family of representations of the algebra  $\mathcal{U}$ . To select some of them, we have to impose an additional condition.

As mentioned above (see (3.14)-(3.15)), in the non-deformed case substitution of the generating matrix  $U_0$  in the formula (2.23) gives the matrix  $L_0$  which exactly coincides with the limit version of the matrix (2.16). It is natural to suppose that the generating matrix  $U$  corresponding to deformed algebra produces in the same way the matrix (2.16) itself. Bearing in mind the property (2.41), we obtain the following

**Proposition 11** *The condition  $\widehat{U} D \widehat{U}^{-1} = L$ , where  $L$  is the matrix (2.16),  $D$  is given by*

$$D = \begin{pmatrix} q^{p/\hbar} & \\ & q^{-p/\hbar} \end{pmatrix} = \begin{pmatrix} q N_1 N_2 & \\ & q^{-1} N_1^{-1} N_2^{-1} \end{pmatrix}, \quad (4.9)$$

and  $\widehat{U}$  is given by (4.8), imposes the following restrictions:

$$\alpha + \beta + \frac{1}{2} = 0, \quad \alpha_0 = q \gamma_0, \quad \beta_0 = \delta_0. \quad (4.10)$$

Substitution of (4.10) into (4.8) completes a description of  $\widehat{U}$  in terms of  $q$ -oscillators.

## B. Connection with $(T^*\mathcal{B})_q$ .

Now we are going to develop another approach to constructing representations of  $\mathcal{U}$ . It is more universal since it is based on the connection (which takes place for arbitrary quantum Lie algebra) of the algebra  $\mathcal{L}$  (see Definition 1) with  $(T^*\mathcal{B})_q$  and on the interpretation of the deformed Borel subgroup  $\mathcal{B}_q$  as a quantum model space.

To clarify the announced connection we start with the following theorem (this is a version of the theorem given in Ref.10 for  $L$ -operators with nonultralocal relations)

**Theorem 3** *Let the matrices  $A$  and  $B$  obey the relations of type (2.1):*

$$R_{\pm} \overset{1}{A} \overset{2}{A} = \overset{2}{A} \overset{1}{A} R_{\pm}, \quad R_{\pm} \overset{1}{B} \overset{2}{B} = \overset{2}{B} \overset{1}{B} R_{\pm} \quad (4.11)$$

and the additional exchange relation

$$\overset{1}{A} \overset{2}{B} = \overset{2}{B} \overset{1}{A} R_+, \quad \overset{2}{A} \overset{1}{B} R_- = \overset{1}{B} \overset{2}{A}. \quad (4.12)$$

*Then the  $L$ -operator constructed by means of similarity transformation*

$$L = A B A^{-1} \quad (4.13)$$

*satisfies the relation (2.3).*

*Remark:* Since (4.11) defines a quantum group structure,  $A^{-1}$  in (4.13) should be understood as an antipode of  $A$ .



*Proof* of Theorem 3 is straightforward :

$$\begin{aligned}
& \overset{1}{L} R_-^{-1} \overset{2}{L} R_- = \overset{1}{A} \overset{1}{B} (\overset{1}{A})^{-1} R_-^{-1} \overset{2}{A} \overset{2}{B} (\overset{2}{A})^{-1} R_- = \overset{1}{A} \overset{1}{B} \overset{2}{A} R_-^{-1} (\overset{1}{A})^{-1} \overset{2}{B} (\overset{2}{A})^{-1} R_- = \\
& = \overset{1}{A} \overset{2}{A} \overset{1}{B} \overset{2}{B} R_+^{-1} (\overset{1}{A})^{-1} (\overset{2}{A})^{-1} R_- = R_+^{-1} \overset{2}{A} \overset{1}{A} \overset{2}{B} \overset{1}{B} R_- (\overset{2}{A})^{-1} (\overset{1}{A})^{-1} = \\
& = R_+^{-1} \overset{2}{A} \overset{2}{B} \overset{1}{A} R_+ (\overset{2}{A})^{-1} \overset{1}{B} (\overset{1}{A})^{-1} = R_+^{-1} \overset{2}{A} \overset{2}{B} (\overset{2}{A})^{-1} R_+ \overset{1}{A} \overset{1}{B} (\overset{1}{A})^{-1} = R_+^{-1} \overset{2}{L} R_+ \overset{1}{L} .
\end{aligned}$$

Thus, for a given quantum group  $G_q$ , the algebra  $\mathcal{L}$  is embedded into the algebra generated by entries of  $A$  and  $B$  obeying (4.11)-(4.12). To argue that (4.11)-(4.12) describe a  $q$ -analogue of  $T^*\mathcal{B}$ , let us notice that the non-symmetric (with respect to  $R$ -matrices) form of the relations (4.12) imposes some restriction on the structure of the matrices  $A$  and  $B$ . Say, if  $R_+$  is an upper triangular matrix, then  $A$  and  $B$  must be upper and lower triangular, respectively. Therefore, one may think of  $A$  and  $B$  as coordinates in the deformed Borel subgroup  $\mathcal{B}_q$  and in the dual quantum space, respectively. In other words, the matrices  $A$  and  $B$  are coordinate and momentum on the deformed phase space  $(T^*\mathcal{B})_q$  respectively. Thus (4.11)-(4.12) may be regarded as a definition of  $(T^*\mathcal{B})_q$  (for additional comments see Ref.10).

We should underline here that, although the matrices  $A$  and  $B$  look similarly on quantum level, they transform into different objects when  $q \rightarrow 1$ . Indeed, in the limit  $q \rightarrow 1$  one has  $L \rightarrow I + \gamma \hbar L_0$  and the corresponding limit forms of  $A$  and  $B$  are

$$A \rightarrow A_0, \quad B \rightarrow I + \gamma \hbar B_0, \quad (4.14)$$

where  $A_0$  is a group-like element, whereas  $B_0$  is rather an element of algebra (see (3.17) as an example of  $A_0$ ,  $B_0$  for  $sl(2)$ ).

Comparing the statements of Theorems 1 and 3 and taking into account the equality (2.41), we get the formula

$$L = A B A^{-1} = \widehat{U} D \widehat{U}^{-1}, \quad (4.15)$$

which points out a possibility to construct the matrix  $\widehat{U}$  obeying (2.35)-(2.38) via the generators of  $(T^*\mathcal{B})_q$ . This connection is very important; below we consider it for  $SL_q(2)$  in all details.

Now let us turn to the example of  $SL_q(2)$ . For  $R_{\pm}$  defined as in (2.8) one can choose

$$A = \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 \\ d & b^{-1} \end{pmatrix}. \quad (4.16)$$

Explicit relations for the generators of  $(T^*\mathcal{B})_q$  following from (4.11)-(4.12) are

$$a c = q^{-1} c a, \quad b c = q^{1/2} c b, \quad a b = q^{1/2} b a; \quad (4.17)$$

$$b d = q^{-1} d b, \quad a d = q^{1/2} d a, \quad c d = q^{-1/2} d c + q^{-1/2} \omega b^{-1} a. \quad (4.18)$$

Performing the following decomposition

$$d = d_0 + d_1 = d_0 + q^{1/2} c^{-1} b^{-1} a, \quad (4.19)$$

we transform (4.18) to homogeneous form:

$$b d_0 = q^{-1} d_0 b, \quad a d_0 = q^{1/2} d_0 a, \quad c d_0 = q^{-1/2} d_0 c. \quad (4.20)$$

Thus, (4.17) and (4.20) describe four variables obeying Weyl-like commutation relations. Using the jargon of conformal field theory, we shall call these formulae "free field representation" and the generators  $a, b, c, d_0$  "free field" variables.

*Remark:* The last of equations (4.18) is nothing but a commutation relation entering the definition of deformed Heisenberg algebra. Indeed, comparing (4.17)-(4.18) with (4.4), one can establish the following correspondence ( $\rho$  stands for arbitrary numerical constant):

$$c \sim N^\rho a^+, \quad d \sim -\omega N^{-1/2-\rho} a, \quad b^{-1} a \sim q^\rho N^{-3/2}.$$

Thus, the transformation (4.19) can be interpreted as "bosonization" of  $q$ -oscillators.

Now, substituting (4.16) in (4.13), we get

$$\begin{aligned} L &= q^{1/2} \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} b & 0 \\ d & b^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & -q c \\ 0 & a \end{pmatrix} = \\ &= \begin{pmatrix} q(b + b^{-1}) + a^{-1} c d_0 & -q^2 a c (b + q a^{-1} c d_0) \\ (a c)^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) & -q^2 a^{-1} c d_0 \end{pmatrix}. \end{aligned} \quad (4.21)$$

This matrix provides a "free field" realization of the algebra  $\mathcal{L}$  for  $U_q(sl(2))$ . Note that the additional scaling factor  $q^{1/2}$  was introduced in (4.21) to ensure a coincidence of the Casimir operators calculated by formulae (2.13) for the matrix (4.21):

$$K_1 = q^2(b + b^{-1}), \quad K_2 = q^3 \quad (4.22)$$

with those for the matrix (2.16). In fact, we redefined the matrix  $B$  in (4.16) as

$$\tilde{B} = q^{1/2} B. \quad (4.23)$$

Comparing the Casimir operator  $K_1$  given by (4.22) with one given by (2.18), we identify the operator  $b$  with the power of the operator of spin  $\hat{j}$ :

$$b = q^{2\hat{j}+1}. \quad (4.24)$$

It follows from (4.22) that matrix  $L$  contains only three independent variables (it is easy to see from the explicit form (4.21) that these are  $b, ac$  and  $a^{-1} c d_0$ ). Moreover, direct calculation using (4.17), (4.20) shows that all elements of the matrix  $L$  commute with operator  $b$ . That agrees with the property (2.25).

Now exploiting the connection described by formula (4.15), one can obtain an exact expression for  $\hat{U}$ .

**Theorem 4** *The algebra  $\mathcal{U} \equiv \{\hat{U}, p\}$  with defining relations (2.35)-(2.38) has the following realization in terms of generators  $a, b, c, d_0$*

$$b = q^{p/\hbar}, \quad \hat{U} = \begin{pmatrix} \frac{1}{\omega} a (b + a^{-1} c d_0) e^{-\frac{i\xi}{2}} & c e^{\frac{i\xi}{2}} \\ \frac{1}{\omega} c^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) e^{-\frac{i\xi}{2}} & a^{-1} e^{\frac{i\xi}{2}} \end{pmatrix}, \quad (4.25)$$

where  $\omega \equiv q - q^{-1}$ ,  $d_0$  is defined in (4.19), and

$$e^{i\xi} = a^{-1} b^\gamma c^{-1} d_0^{-1} \quad (4.26)$$

with  $\gamma$  being an arbitrary constant.

This theorem gives a "free field" representation of the algebra  $\mathcal{U}$ . Let us remark that the remaining freedom in (4.26) corresponds only to canonical transformations (since  $\xi$  and  $p$  are conjugate variables).

The formulated theorem will be proved in several steps. First, we introduce a lower-triangular matrix which diagonalizes the matrix  $\tilde{B}$ :

$$V = \begin{pmatrix} v_1 & 0 \\ v_3 & v_2 \end{pmatrix}, \quad \tilde{B} = V \tilde{B}_0 V^{-1}, \quad \tilde{B}_0 = \begin{pmatrix} q^{1/2} b & 0 \\ 0 & q^{1/2} b^{-1} \end{pmatrix} \equiv q^{1/2} B_0. \quad (4.27)$$

**Proposition 12** *A possible solution for the matrix  $V$  is*

$$v_1 = v_1(b), \quad v_2 = v_2(b), \quad v_3 = d v_1(b) f(b), \quad (4.28)$$

where  $v_1(b)$ ,  $v_2(b)$  are arbitrary functions of  $b$  and  $f(b) = (b - qb^{-1})^{-1}$ .

Thus, matrix  $L$  given by (4.21) admits a decomposition of the form:

$$L = \hat{U}_0 \tilde{B}_0 \hat{U}_0^{-1}, \quad \hat{U}_0 = A V. \quad (4.29)$$

However, this diagonalization is not unique. Using an arbitrary power of the diagonal matrix  $Q$ , which depends on the variable conjugate to  $b$ ,

$$Q = \begin{pmatrix} e^{i\xi} & \\ & e^{-i\xi} \end{pmatrix}, \quad b e^{i\xi} = q e^{i\xi} b, \quad (4.30)$$

we obtain a family of diagonalizing matrices:

$$L = \hat{U}_\delta \tilde{B}_\delta \hat{U}_\delta^{-1}, \quad \hat{U}_\delta = A V Q^\delta, \quad \tilde{B}_\delta = Q^{-\delta} \tilde{B}_0 Q^\delta = q^\delta \tilde{B}_0 = q^{\delta+1/2} B_0. \quad (4.31)$$

An explicit form of the diagonalizing matrix is

$$\hat{U}_\delta = A V Q^\delta = \begin{pmatrix} (a v_1 + c d v_1 f) e^{i\delta\xi} & c v_2 e^{-i\delta\xi} \\ a^{-1} d v_1 f e^{i\delta\xi} & a^{-1} v_2 e^{-i\delta\xi} \end{pmatrix}. \quad (4.32)$$

Here we should describe a new object  $e^{i\xi}$  which appeared in the matrix  $\hat{U}$ . We assume that the following Weyl-like relations hold:

$$a e^{i\xi} = q^\alpha e^{i\xi} a, \quad b e^{i\xi} = q e^{i\xi} b, \quad c e^{i\xi} = q^\beta e^{i\xi} c, \quad d_0 e^{i\xi} = q^\gamma e^{i\xi} d_0. \quad (4.33)$$

**Proposition 13** *The set of equations (4.33) is equivalent to*

$$e^{i\xi} = a^{\beta+(\gamma-1)/2} b^\gamma c^{(\gamma-1)/2-\alpha} d_0^{-1}. \quad (4.34)$$

Now we have to remind that the matrix  $U$  (and  $\widehat{U}$  as well) described in the Theorem 1 has to satisfy the relation (2.20) or, equivalently, to the relation

$${}^1_{B_0} \widehat{U}_\delta = {}^2_{\widehat{U}_\delta} {}^1_{B_0} \sigma, \quad (4.35)$$

where  $\sigma$  and  $B_0$  were introduced in (2.20) and (4.27), respectively. A straightforward calculation using (4.17)-(4.18) leads to the following.

**Proposition 14** *The matrix  $\widehat{U}_\delta$  given by (4.32) satisfies the relation (4.35) only for  $\delta = -1/2$ .*

It is worth mentioning that such a choice of  $\delta$  exactly compensates the renormalization of the matrix  $B$  in (4.23), i.e.,  $\widetilde{B}_{-1/2} = B_0$ .

Bearing in mind the formula (4.19), one can rewrite (4.32) for  $\delta = -\frac{1}{2}$  as follows

$$\widehat{U} \equiv \widehat{U}_{-1/2} = \begin{pmatrix} a(b + a^{-1} c d_0) w e^{-\frac{i\xi}{2}} & c v e^{\frac{i\xi}{2}} \\ c^{-1}(b^{-1} + q^{-1} a^{-1} c d_0) w e^{-\frac{i\xi}{2}} & a^{-1} v e^{\frac{i\xi}{2}} \end{pmatrix}, \quad (4.36)$$

where  $w \equiv f(b)v_1(b)$ ,  $v \equiv v_2(b)$ .

Finally, a direct check shows (see the Appendix C) that the matrix (4.36) obeys eqs. (2.35)-(2.38) if the functions  $w, v$  are constant (we chose them as follows:  $v(b) = 1$ ,  $w(b) = \frac{1}{w}$ ) and the coefficients in (4.33)-(4.34) satisfy the conditions  $\beta = -\alpha$ ,  $\gamma = \alpha - \beta - 1 = 2\alpha - 1$ . Thus, Theorem 4 is proven.

Let us end the discussion of relation of  $(T^*\mathcal{B})_q$  to algebras  $\mathcal{L}$  and  $\mathcal{U}$  with one more statement:

**Theorem 5** *The algebra generated by coordinates on  $(T^*\mathcal{B})_q$  is isomorphic to the algebra generated by entries of the matrix  $L$  and  $Q$ .*

*Proof.* Formulae (4.21) and (4.26) provide explicit expressions for entries of  $L$  and  $Q$  via the generators  $a, b, c, d_0$  (up to unessential canonical transformation in (4.26)). Conversely, suppose matrix  $L$  and the element  $e^{i\xi}$  are given. Then, as it follows from (4.21), one can construct from entries of  $L$  the combinations  $b, ac$  and  $a^{-1}cd_0$ . Together with (4.26) this allows to recover the “coordinates”  $a, b, c, d_0$ .

Although we considered this theorem only for the case of  $SL_q(2)$ , there is an evidence that it holds for the generic case. Indeed, in the case of  $G_q = SL_q(N)$  a point on the quantum bundle  $(T^*\mathcal{B})_q$  is parameterized by  $N \times N$  matrices  $A$  and  $B$ . As above, the matrix  $L = ABA^{-1}$  satisfies (2.3) and therefore its entries generate the corresponding algebra  $\mathcal{L}$ . However, the dimension of  $(T^*\mathcal{B})_q$  exceeds the dimension of  $\mathcal{L}$ :  $\dim(T^*\mathcal{B})_q - \dim \mathcal{L} = (N^2 + N - 2) - (N^2 - 1) = N - 1$ . It is very probable that the remaining  $(N - 1)$  generators are exactly those that enter the diagonal unimodular  $N \times N$  matrix  $Q$ .

### C. Explicit representation.

Now we face the problem of constructing of an explicit representation for the generators  $a, b, c, d_0$ . A Weyl-like form of the commutation relations (4.17),(4.20) points out the

possibility of getting a realization for these generators in terms of two pairs of canonical variables. This also means (due to the interpretation of (4.19) as "bosonization" of  $q$ -oscillators) that the generators  $a, b, c, d$  admit a realization via  $q$ -oscillators. Evidently, such a representation is not unique.

It is natural to realize  $a, b, c, d_0$  as operators acting on the  $q$ -analogue of the space  $D(z_1, z_2)$ . We shall denote this space as  $D_q(z_1, z_2)$ . The space  $D_q(z_1, z_2)$  is spanned on the basic vectors of form (remember that  $[x]$  stands for  $q$ -numbers)

$$|j, m\rangle = \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{[j+m]! [j-m]!}}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad m = -j, \dots, j. \quad (4.37)$$

One can define on  $D_q(z_1, z_2)$  such a scalar product that the system (4.37) is orthonormal, that is  $\langle j, m | j', m' \rangle = \delta_{jj'} \delta_{mm'}$ .

*Remark:* This scalar product is a deformation of (3.11). Its explicit form makes use of the  $q$ -exponent and the Jackson integral. See Ref.20 for details.

In all formulae concerning the space  $D_q(z_1, z_2)$  we suppose that  $q$  is chosen as described in Sec.II (i.e., it belongs either to the real axis or to the unit circle at the complex plane). In this case an analogue of the rule of conjugation (3.12) is

$$(z_i)^* = z_i^{-1} [z_i \partial_i], \quad (z_i \partial_i)^* = z_i \partial_i. \quad (4.38)$$

The formulae (4.24) and (4.25) imply that the generator  $b$  is a power of the operator of spin. Hence, on the space  $D_q(z_1, z_2)$  it is given by

$$b = q^{z_1 \partial_1 + z_2 \partial_2 + 1} = q N_1 N_2. \quad (4.39)$$

Next, let us remind that we already know the limit versions of the generators  $a, b, c, d$  (see Proposition 7; one should take into account the rescaling (4.23)). Their appropriate deformations for generic  $q$  are described by

**Proposition 15** *The set of operators (with arbitrary constants  $\lambda_i, \nu_i$ )*

$$\begin{aligned} a &= q^{\lambda_0} z_1^{-1/2} N_1^{\lambda_1}, \quad c = q^{\nu_0} z_1^{-1/2} z_2 N_1^{\lambda_1-2} N_2^{\nu_2}, \\ b &= q N_1 N_2, \quad d = -q^{\lambda_0-\nu_0+\nu_2} z_2^{-1} (N_2 - N_2^{-1}) N_1 N_2^{-\nu_2} \end{aligned} \quad (4.40)$$

*satisfies (4.17)-(4.18) and gives in the limit  $q \rightarrow 0$  the generators found in (3.17).*

Although due to Theorem 4 this proposition gives a family of representations for  $\mathcal{U}$ , we again should impose an additional condition using the matrix (2.16) as a standard (justification for this trick was given above).

**Proposition 16** *Matrix  $L$  given by (4.21) coincides with the matrix (2.16) taken in the representation (4.6) provided that*

$$b = q N_1 N_2, \quad a c = q^{-1/2} z_1^{-1} z_2 N_1^{-1/2} N_2^{-1/2}, \quad a^{-1} c d_0 = -N_1^{-1} N_2. \quad (4.41)$$

Comparing the statements of Propositions 15 and 16, we derive:

$$\begin{aligned} a &= q^{\lambda_0} z_1^{-1/2} N_1^{3/4}, & b &= q N_1 N_2, \\ c &= q^{\nu_0} z_1^{-1/2} z_2 N_1^{-5/4} N_2^{-1/2}, \\ d_0 &= -q^{\lambda_0 - \nu_0 - 1/2} z_2^{-1} N_1 N_2^{3/2}, & q^{\lambda_0 + \nu_0} &= q^{-1/8}. \end{aligned} \quad (4.42)$$

Substituting (4.42) into (4.26) (and remember that (4.26) is defined only up to a coefficient), we get

$$e^{i\xi} = q^{2\epsilon} z_1 N_1^{\gamma-1/2} N_2^{\gamma-1}, \quad (4.43)$$

where  $\gamma$  and  $\epsilon$  are arbitrary. Finally, substituting (4.42)-(4.43) into (4.36), we obtain (one should remember  $U$  and  $\hat{U}$  are defined only up to arbitrary scaling factor)

$$\hat{U} = \begin{pmatrix} \frac{1}{\omega} \alpha_0 z_1^{-1} N_1^{1-\gamma/2} N_2^{3/2-\gamma/2} (N_1 - N_1^{-1}) & \beta_0 z_2 N_1^{\gamma/2-3/2} N_2^{\gamma/2-1} \\ -\frac{1}{\omega} q^{-1} \alpha_0 z_2^{-1} N_1^{1/2-\gamma/2} N_2^{1-\gamma/2} (N_2 - N_2^{-1}) & \beta_0 z_1 N_1^{\gamma/2-1} N_2^{\gamma/2-1/2} \end{pmatrix}. \quad (4.44)$$

It is easy to check that the family of matrices (4.44) exactly coincides with what was obtained in  $q$ -oscillator approach (see Propositions 10 and 11).

#### D. Quantum Clebsch-Gordan coefficients.

Using the connection formula (2.34) we get from (4.44) a family of matrices  $U$  which provide possible representations of the algebra  $\mathcal{U}$ . It is natural to study an action of the entries of these matrices on the space  $D_q(z_1, z_2)$  described above. On the basic vectors (4.37) these operators act as follows:

$$\begin{aligned} U_1 |j, m\rangle &= C_1 q^{\frac{1}{2}(j-m+1)} \sqrt{\frac{[j+m]}{[2j+1]}} |j - \frac{1}{2}, m - \frac{1}{2}\rangle, \\ U_2 |j, m\rangle &= C_2 q^{-\frac{1}{2}(j+m)} \sqrt{\frac{[j-m+1]}{[2j+1]}} |j + \frac{1}{2}, m - \frac{1}{2}\rangle, \\ U_3 |j, m\rangle &= -C_3 q^{-\frac{1}{2}(j+m+1)} \sqrt{\frac{[j-m]}{[2j+1]}} |j - \frac{1}{2}, m + \frac{1}{2}\rangle, \\ U_4 |j, m\rangle &= C_4 q^{\frac{1}{2}(j-m)} \sqrt{\frac{[j+m+1]}{[2j+1]}} |j + \frac{1}{2}, m + \frac{1}{2}\rangle, \end{aligned} \quad (4.45)$$

where the coefficients  $C_i$  do not depend on  $m$ .

Note that, similarly to the classical case, the operators  $U_i$  correspond to the basic shifts on the model space. Comparing the matrix elements  $\langle j', m' | U_i | j, m \rangle$  following from (4.45) with values of CGC for  $U_q(sl(2))$  given by  $q$ -analogue of the Van-der-Waerden formula,<sup>18,21</sup> which for the decomposition of  $V_j \otimes V_{1/2}$  looks like following

$$\begin{aligned} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & m' & m'' \end{array} \right\}_q &= \delta_{m'', m+m'} \left( \frac{[j + \frac{1}{2} - j'']! [j + j'' - \frac{1}{2}]! [j'' + \frac{1}{2} - j]!}{[j + j'' + \frac{3}{2}]!} \right)^{1/2} \times \\ &\times q^{\frac{1}{2}(j+\frac{1}{2}-j'')(j+j''+\frac{3}{2})+jm'-\frac{1}{2}m} \times \end{aligned} \quad (4.46)$$

$$\times \sum_{r \geq 0} \frac{(-1)^r q^{-r(j+j''+\frac{3}{2})} ([j+m]![j-m]![j''+m'']![j''-m'']![2j''+1])^{1/2}}{[r]![j+\frac{1}{2}-j''-r]![j-m-r]![\frac{1}{2}+m'-r]![j''-\frac{1}{2}+m+r]![j''-j-m'+r]!},$$

we establish the following correspondence

$$\begin{aligned} \langle j'', m'' | U_1 | j, m \rangle &= \delta_{j'', j-\frac{1}{2}} \alpha_0 q^{(1-\gamma/2)j-1/2} \left\{ \begin{matrix} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{matrix} \right\}_q, \\ \langle j'', m'' | U_2 | j, m \rangle &= \delta_{j'', j+\frac{1}{2}} \beta_0 q^{(\gamma/2-1)j} \left\{ \begin{matrix} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{matrix} \right\}_q, \\ \langle j'', m'' | U_3 | j, m \rangle &= \delta_{j'', j-\frac{1}{2}} \alpha_0 q^{(1-\gamma/2)j-1/2} \left\{ \begin{matrix} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{matrix} \right\}_q, \\ \langle j'', m'' | U_4 | j, m \rangle &= \delta_{j'', j+\frac{1}{2}} \beta_0 q^{(\gamma/2-1)j} \left\{ \begin{matrix} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{matrix} \right\}_q. \end{aligned}$$

Thus we derive an analogue of Proposition 8:

**Proposition 17** *The generators  $U_i$  of the algebra  $\mathcal{U}$  are operators of the basic shifts on the model space for  $U_q(sl(2))$  and they generate the  $q$ -Clebsch-Gordan coefficients corresponding to decomposition of the product  $V_j \otimes V_{\frac{1}{2}}$  of irreps of  $U_q(sl(2))$ .*

*Remark:* Putting  $\alpha_0 = q^{1/2}$ ,  $\beta_0 = 1$  and  $\gamma = 2$  in (4.44), we get the following generating matrix

$$U = \begin{pmatrix} z_1^{-1} [z_1 \partial_1] q^{\frac{1}{2}(z_2 \partial_2 + 1)} & z_2 q^{-\frac{1}{2} z_1 \partial_1} \\ -z_2^{-1} [z_2 \partial_2] q^{-\frac{1}{2}(z_1 \partial_1 + 1)} & z_1 q^{\frac{1}{2} z_2 \partial_2} \end{pmatrix} \frac{1}{\sqrt{[p/\hbar]}}, \quad \text{Det} U = q^{1/2}, \quad (4.47)$$

which may be called “exact” as it satisfies (4.45) with  $C_i = 1$ . The question about unitarity of the matrix (4.47) is discussed in Appendix B.

### E. Generalized Wigner-Eckart theorem.

As we demonstrated in the previous section, entries of the matrix  $U_0$  are tensor operators of spin 1/2 for  $\mathcal{J} = sl(2)$ , hence they provide a realization of the Wigner-Eckart theorem. Let us now consider the matrix  $U$  from this point of view.

The theory of tensor operators for quantum algebras was discussed by many authors (see, e.g., Ref.22). In particular, generalized Wigner-Eckart theorem (in the case of  $\mathcal{J}_q = U_q(sl(2))$ ) reads as follows.

**Theorem 6** *Let  $l_+$ ,  $l_-$  and  $l_3$  be the generators of  $U_q(sl(2))$  and let  $T_m^j$ ,  $m = -j, \dots, j$  be a set of operators acting on the deformed model space  $\mathcal{M}$  and obeying the commutation relations*

$$[l_3, T_m^j] = m T_m^j, \quad l_{\pm} T_m^j q^{l_3} - q^{l_3 \mp 1} T_m^j l_{\pm} = \sqrt{[j \mp m][j \pm m + 1]} T_{m \pm 1}^j. \quad (4.48)$$

*Then the matrix elements of  $T_m^j$  on  $\mathcal{M}$  are proportional to  $q$ -Clebsch-Gordan coefficients:*

$$\langle j'' m'' | T_m^j | j' m' \rangle = C_{j'' j'}^j \left\{ \begin{matrix} j' & j & j'' \\ m' & m & m'' \end{matrix} \right\}_q,$$

*where the coefficients  $C_{j'' j'}^j$  do not depend on  $m, m', m''$ .*

Proposition 17 implies that  $U_i$  may be regarded as  $q$ -tensor operators. Indeed, using (4.44) and (4.6), one can check that  $U_i$  satisfy (4.48) (one obtains for  $U_i$  deformations of relations (3.23)). Similarly to the classical case we have the following.

**Proposition 18** *The generators  $U_i$  of the algebra  $\mathcal{U}$  form a basis for  $q$ -tensor operators of spin  $1/2$ , that is components  $T_{1/2}^{1/2}$  and  $T_{-1/2}^{1/2}$  of any  $q$ -tensor operator of spin  $1/2$  can be realized as linear combinations of  $U_i$ :*

$$T_{-1/2}^{1/2} = \mu(p) U_1 + \nu(p) U_2, \quad T_{1/2}^{1/2} = \mu(p) U_3 + \nu(p) U_4, \quad (4.49)$$

where  $\mu(p)$  and  $\nu(p)$  are functions only of  $p$ .

*Remark:* Unlike the classical case, solution (4.44) gives a family of matrices  $U$ . However, the corresponding matrix elements  $\langle j''m''|U_i|j'm' \rangle$  differ only by factors which do not depend on  $m'$ ,  $m''$ . Thus, any representative of obtained family of matrices  $U$  may be used in Proposition 18.

Let us end the description of the algebra  $\mathcal{U}$  from the point of view of theory of  $q$ -tensor operators with the following statement:

**Proposition 19** *The matrices  $U$  and  $L$  defined in the Theorem 1 obey the relation*

$$R_- \overset{1}{U} \overset{2}{L} = \overset{2}{L} R_+ \overset{1}{U}. \quad (4.50)$$

The proof is straightforward

$$\begin{aligned} R_- \overset{1}{U} \overset{2}{L} &= R_- \overset{1}{U} \overset{2}{U} \overset{2}{D} (\overset{2}{U})^{-1} = \overset{2}{U} \overset{1}{U} \mathcal{R}_- \overset{2}{D} (\overset{2}{U})^{-1} = \\ &= \overset{2}{U} \overset{1}{U} \overset{2}{D} \sigma \mathcal{R}_+ (\overset{2}{U})^{-1} = \overset{2}{U} \overset{2}{D} \overset{1}{U} \mathcal{R}_+ (\overset{2}{U})^{-1} = \overset{2}{U} \overset{2}{D} (\overset{2}{U})^{-1} R_+ \overset{1}{U} = \overset{2}{L} R_+ \overset{1}{U}; \end{aligned}$$

it makes use the relations (2.20)-(2.21) and the property (2.24).

A remarkable fact is that (4.50) may be used for definition of  $q$ -tensor operators instead of (4.48). Indeed, in the limit  $\gamma \rightarrow 0$  it turns into

$$[\overset{1}{U}_0, \overset{2}{L}_0] = \Lambda \overset{1}{U}_0, \quad \Lambda = \begin{pmatrix} 1/2 & & & \\ & -1/2 & 1 & \\ & 1 & -1/2 & \\ & & & 1/2 \end{pmatrix}. \quad (4.51)$$

Using the explicit form of  $L_0$  given in (3.14), one can easily check that this matrix relation is equivalent to (3.23). More on  $R$ -matrix description of  $q$ -tensor operators is given in Ref.23.

## CONCLUSION.

In this paper we have constructed the  $q$ -analogue of the phase space  $T^*\mathcal{B}$  and clarified its role in description of the model representation of the corresponding quantum group  $G_q$ . We unraveled a connection between the algebras generating by entries of matrix  $(A, B)$ ,  $(U, D)$  and  $(L, Q)$ . The general formulae were concretized by the example of  $G = SL(2)$ .



An extension of the described scheme to the case of arbitrary group  $G$  will definitely improve understanding of the role played by the matrix  $\mathcal{R}(p)$  which so far has been discussed in the literature much less than the standard matrix  $R$ .

The results of this paper can be generalized in several directions even for the case of  $SL(2)$ . The first is consideration of the matrix  $U$  with an auxiliary space corresponding to the higher spin representation. It must lead to an exact form of generating matrix for all CGC. The work in this direction is in progress now. The second point to be discussed is the case of  $q$  being a root of unity. The structure of  $\mathcal{R}(p)$  allows to hope that reduction on so-called "good" representations will be quite natural in our formalism. However, this case is to be examined more carefully.

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### Appendix A: Proof of Theorem 1.

Using (2.20)-(2.21) together with the identity (2.24) and taking into account that matrices  $\overset{1}{D}$ ,  $\overset{2}{D}$  and  $\sigma$  mutually commute, we check

$$\begin{aligned}
& \overset{1}{L} R_-^{-1} \overset{2}{L} R_- = \overset{1}{U} \overset{1}{D} (\overset{1}{U})^{-1} R_-^{-1} \overset{2}{U} \overset{2}{D} (\overset{2}{U})^{-1} R_- = \\
& = \overset{1}{U} \overset{1}{D} \overset{2}{U} \mathcal{R}_-^{-1}(p) (\overset{1}{U})^{-1} \overset{2}{D} (\overset{2}{U})^{-1} R_- = \overset{1}{U} \overset{2}{U} \overset{1}{D} \sigma \mathcal{R}_-^{-1}(p) \overset{2}{D} \sigma (\overset{1}{U})^{-1} (\overset{2}{U})^{-1} R_- = \\
& = R_+^{-1} \overset{2}{U} \overset{1}{U} \mathcal{R}_+(p) \overset{1}{D} \sigma \mathcal{R}_-^{-1}(p) \overset{2}{D} \sigma \mathcal{R}_-(p) (\overset{2}{U})^{-1} (\overset{1}{U})^{-1} = \\
& = R_+^{-1} \overset{2}{U} \overset{1}{U} \overset{2}{D} \sigma \overset{1}{D} \mathcal{R}_-(p) (\overset{2}{U})^{-1} (\overset{1}{U})^{-1} = R_+^{-1} \overset{2}{U} \overset{2}{D} \overset{1}{U} \mathcal{R}_+(p) \sigma \overset{1}{D} (\overset{2}{U})^{-1} (\overset{1}{U})^{-1} = \\
& = R_+^{-1} \overset{2}{U} \overset{2}{D} \overset{1}{U} \mathcal{R}_+(p) (\overset{2}{U})^{-1} \overset{1}{D} (\overset{1}{U})^{-1} = R_+^{-1} \overset{2}{U} \overset{2}{D} (\overset{2}{U})^{-1} R_+ \overset{1}{U} \overset{1}{D} (\overset{1}{U})^{-1} = R_+^{-1} \overset{2}{L} R_+ \overset{1}{L}.
\end{aligned}$$

### Appendix B: On conjugation of $U_0$ and $U$ .

First we consider the matrix  $U_0$ . Using the rules of conjugation (3.12) (and taking into account that  $p^* = p$ ), one can check that the matrix conjugated to  $U_0$  does not coincide with  $U_0^{-1}$ ; that is, the matrix  $U_0$  itself is not unitary. However, it turns out that the transposed matrix (one should remember that in general  $(U^T)^{-1} \neq (U^{-1})^T$  for matrices with non-commuting entries)

$$U_0^T = \begin{pmatrix} \partial_1 & -\partial_2 \\ z_2 & z_1 \end{pmatrix} \sqrt{\frac{\hbar}{p}}$$

satisfies the unitarity condition:

$$(U_0^T)^* = \sqrt{\frac{\hbar}{p}} \begin{pmatrix} z_1 & \partial_2 \\ -z_2 & \partial_1 \end{pmatrix} = (U_0^T)^{-1}.$$

In the deformed case (recall that  $q$  can be either real or  $|q| = 1$ ) the matrix  $U$  includes the operator  $N$  which conjugates in different ways for the different choices of  $q$ . Let us consider the matrix  $U$  given by (4.47). The conjugated matrix can be constructed according to the rules (4.38). Using the formula (4.1), one can check that the unitarity condition  $(U^T)^* U^T = U^T (U^T)^* = I$  (i.e., the same as in the non-deformed case) for the transposed matrix holds only for real  $q$ . For  $|q| = 1$  see Ref.13.

### Appendix C: Proof of Theorem 4.

Here we complete the proof of Theorem 4, i.e., we have to prove that matrix (4.36) satisfies (2.35)-(2.37) if the following conditions

$$\alpha + \beta = 0, \quad \gamma + \beta - \alpha + 1 = 0, \quad v(b) = 1, \quad w(b) = 1/\omega \quad (\text{C } 1)$$

are fulfilled.

First, using relations (4.17), (4.20), (4.33) and conditions (C 1), we check

$$\begin{aligned} \widehat{U}_1 \widehat{U}_2 &= a(b + a^{-1} c d_0) e^{-i\xi} c e^{i\xi} = q^{-1/2+\beta/2} c a(b + q a^{-1} c d_0) e^{i\xi} e^{-i\xi} = \\ &= q^{\alpha/2+\beta/2} c e^{i\xi} a(b + q^{(\gamma+\beta-\alpha+1)/2} a^{-1} c d_0) e^{-i\xi} = \widehat{U}_2 \widehat{U}_1. \end{aligned}$$

$$\begin{aligned} \widehat{U}_1 \widehat{U}_3 &= a(b + a^{-1} c d_0) e^{-i\xi} c^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) e^{-i\xi} = \\ &= q^{1/2-\beta/2} c^{-1} a(b + q^{-1} a^{-1} c d_0) e^{-i\xi} (b^{-1} + q^{-1} a^{-1} c d_0) e^{-i\xi} = \\ &= q^{-\beta/2} c^{-1} a(b^{-1} + q^{-1} a^{-1} c d_0) (b + q^{-1} a^{-1} c d_0) e^{-i\xi} e^{-i\xi} = \\ &= q^{-1/2-\beta/2} c^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) a(b + q^{-1} a^{-1} c d_0) e^{-i\xi} e^{-i\xi} = \\ &= q^{-1-\alpha/2-\beta/2} c^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) a e^{-i\xi} (b + a^{-1} c d_0) e^{-i\xi} = q^{-1} \widehat{U}_3 \widehat{U}_1. \end{aligned}$$

The rest of relations (2.35) can be proved similarly.

Next, note that relation (2.36) can be rewritten as follows

$$\widehat{U}_1 \widehat{U}_4 (b - b^{-1}) - \widehat{U}_4 \widehat{U}_1 (q b - q^{-1} b^{-1}) = -\omega \widehat{U}_3 \widehat{U}_2 b. \quad (\text{C } 2)$$

To prove this equality we transform its l.h.s. and r.h.s. as follows

$$\begin{aligned} \widehat{U}_1 \widehat{U}_4 (b - b^{-1}) - \widehat{U}_4 \widehat{U}_1 (q b - q^{-1} b^{-1}) &= a(b + a^{-1} c d_0) e^{-i\xi} a^{-1} e^{i\xi} (b - b^{-1}) - \\ &- a^{-1} e^{i\xi} a(b + a^{-1} c d_0) e^{-i\xi} (q b - q^{-1} b^{-1}) = q^{-\alpha/2} (q^{1/2} b + q^{-1/2} a^{-1} c d_0) (b - b^{-1}) - \\ &- q^{-\alpha/2} (q^{-1/2} b + q^{1/2} a^{-1} c d_0) (q b - q^{-1} b^{-1}) = -q^{-\alpha/2} \omega (q^{1/2} a^{-1} c d_0 b + q^{-1/2}); \\ \widehat{U}_3 \widehat{U}_2 b &= c^{-1} (b^{-1} + q^{-1} a^{-1} c d_0) e^{-i\xi} c e^{i\xi} b = \\ &= q^{\beta/2} (q^{-1/2} b^{-1} + q^{1/2} a^{-1} c d_0) b = q^{\beta/2} (q^{-1/2} + q^{1/2} a^{-1} c d_0 b). \end{aligned}$$

Thus, the equality (C 2) is fulfilled if conditions (C 1) are valid. The relation (2.37) can be proved in the same way.

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